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Kelvin–Helmholtz Instability of Miscible Ferrofluids

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We study the stability of an interface between two inviscid magnetic fluids of different densities flowing parallel to each other in an oscillatory manner. The system is pervaded by a uniform oblique magnetic field distribution. The analysis allows for mass and heat transfer across the interface. A general eigenvalue relation is derived and discussed analytically. The classical stability criterion is found to be substantially modified due to the effect of the oblique magnetic field with mass and heat transfer. Some previous studies are reported for appropriate data choices. The longitudinal magnetic field has a strong stabilizing influence on all wavelengths, which can be used to suppress the destabilizing influence of the mass and heat transfer. We conclude with a discussion of the stability of unsteady shear layers on the basis of the results. The parametric excitation of the surface waves is analyzed by means of the multiple-time-scales method. The transition curves are obtained analytically.

1. INTRODUCTION

When a body of fluid underlies a denser one, a gravitational instability results. This is commonly termed Rayleigh–Taylor instability, where the two fluids are separated by a horizontal boundary. The theoretical and experimental properties of such instabilities have been used to model a number of geophysical processes. Many of the geophysical processes of interest could be better modeled by the introduction of a gravitationally stable fluid beneath the buoyant layer. Wilcock and Whitehead (1991) reported both theoretical and experimental results for the instability developed by a three-layer system comprising a thin layer of buoyant low-viscosity fluid sandwiched between two thick layers of equal properties. The initial stages of the Rayleigh–Taylor instability can be analyzed theoretically using linearized flow equations. The

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solutions can be obtained for the growth rate of instabilities as a function of wavelength and other parameters of the system (Chandrasekhar, 1961).

When different layers of a stratified fluid are in horizontal motion, we get another type of instability. The instability of the plane interface between two superposed fluids with a relative horizontal velocity is called Kelvin–Helmholtz instability. Kelvin–Helmholtz instability is important in understanding a variety astrophysical phenomena involving sheared plasma flow (e.g., the stability of the solar wind–magnetosphere interface, interaction between adjacent streams of different velocities in the solar wind, and dynamo generation of cosmic magnetism). The linear development of Kelvin–Helmholtz-instability magnetic fluids has been carried out by Rosensweig (1985). His analysis revealed that the velocity difference that can be supported by the fluids before the instability sets in is enhanced if the difference in the permeabilities of the fluids across the interface and the strength of the applied magnetic field are increased. Walker *et al.* (1993) presented a modified Kelvin–Helmholtz instability model that predicts that an instability can occur at the free surface of a fluid with a nonuniform velocity.

The effect of a magnetic field on the stability of a steady flow field has received considerable attention (see, for example, Chandrasekhar, 1961). The effect of an unsteady flow is of some interest; it was studied by Roberts (1973), who considered the coupling between the oscillatory component in the basic flow and the effect of the magnetic field. The effect of an oscillatory magnetic field (in addition to a steady component) on a steady velocity distribution has been investigated by Drazin (1961) with the intention of determining whether such an oscillatory field is more stabilizing than the corresponding steady one. The nonlinear wave propagation on the interface between two superposed fluids acted upon by a tangential periodic magnetic field has been investigated by El-Dib (1993), who used the multiple-time-scales method. His stability analysis reveals the existence of both resonant and nonresonant cases.

In all the work cited above, the stability was studied when the interfacial transfer of mass and heat was negligible. The mass transfer across the interface due to evaporation and condensation processes would play an important role. The transfer of mass and heat across an interface is important for solving many problems in chemical engineering and geophysics. The effect of mass and heat transfer across an interface on the motion of fluids was given by Hsieh (1978), who formulated the general problem of interfacial fluid flow with mass and heat transfer in plane geometry and applied it to discuss the stability of both the Rayleigh–Taylor and the Kelvin–Helmholtz models in the problem of film-boiling heat transfer. Nayak and Chakraborty (1984) studied the problem of Kelvin–Helmholtz stability with mass and heat transfer but in cylindrical geometry, using Hsieh's simplified formulation and compared the results with those in plane geometry. Recently, Moatimid (1994) studied the effect of a periodic electric field with mass and heat transfer on the Rayleigh–Taylor instability. His analysis resulted in a Mathieu equation.

In this work, we investigate the stability of an unsteady basic flow of inviscid ferrofluids. The analysis allows for mass and heat transfer across the interface. The fluids are pervaded by a uniform oblique magnetic field distribution. The corresponding steady problem is the classic one of Kelvin–Helmholtz instability. We show that the oscillations of the basic flow can cause a parametric resonance. The analysis results in an ordinary differential equation with periodic coefficients which are analyzed by means of the method of multiple-time-scales. The analysis reveals the existence of both resonant and nonresonant cases. The transition curves are obtained.

2. THE GOVERNING BASIC EQUATIONS

We wish to consider the stability of a flow which is dependent upon time and also satisfies the basic equations of motion. Consider the parallel flow of two incompressible, inviscid magnetic fluids confined between two parallel rigid planes $z = -h_1$ and $z = h_2$. Let x and y be the coordinates in the plane of the interface; $z = \eta(x, y, t)$, where z is the coordinate normal to the interface; and u, v, and w are the corresponding velocity components. At the equilibrium state, the interface is taken to be $z = 0$. Fluid 1 occupies the region $-h_1 < z < \eta$, while fluid 2 occupies the region $\eta < z < h_2$. The densities of the lower and upper fluids are $\rho^{(1)}$ and $\rho^{(2)}$, respectively. The motion is assumed to be irrotational under gravity $\mathbf{g}(0, 0, -g)$. Let the velocities of the basic flow be $U^{(j)}(t)$, where $j = 1, 2$ refers, respectively, to the lower or upper fluids. The temperatures at $z = -h_1$, $z = h_2$, and $z = 0$ are kept at, respectively, T_1, T_2, and T_0, so that $T_1 > T_0 > T_2$.

Consider that the fluids are pervaded by a uniform oblique magnetic fluid $\mathbf{H}^{(j)} = (H_0, 0, H_*^{(j)})$. We shall assume that there are no free currents at the surface of separation in the equilibrium state, and therefore the magnetic induction is continuous at the interface, i.e., $\mu^{(1)}H_*^{(1)} = \mu^{(2)}H_*^{(2)}$, where $\mu^{(1)}$ and $\mu^{(2)}$ are the lower and the upper magnetic permeabilities, respectively.

For an irrotational flow, an integral of the equations of motion may be written as

$$\frac{\pi^{(j)}}{\rho^{(j)}} + \frac{1}{2}(u^{(j)2} + v^{(j)2} + w^{(j)2}) + gz + \frac{\partial \varphi^{(j)}}{\partial t} = C^{(j)}(t) \tag{2.1}$$

where π is the pressure, $\mathbf{V}^{(j)} = (u^{(j)}, v^{(j)}, w^{(j)})$ is the velocity field, φ is the velocity potential, and $C^{(j)}(t)$ is an arbitrary integration function of time, which may be taken as

$$C^{(j)}(t) = \rho^{(j)}U^{(j)^2}/2 - \tfrac{1}{2}\mu^{(j)}H_0^2 + \tfrac{1}{2}\mu^{(j)}H_*^{(j)^2} \tag{2.2}$$

In order to have a parallel basic flow, the interface must be plane, and the pressure must balance across it. By suitable choice of $C^{(j)}(t)$, one can balance all terms except the terms involving the $\varphi^{(j)}$, which depend upon the x direction as

$$\varphi^{(j)} = xU^{(j)}(t) \tag{2.3}$$

However, the difference in pressure can be set equal to zero if

$$\frac{dU^{(1)}}{dt} = \frac{\rho^{(2)}}{\rho^{(1)}}\frac{dU^{(2)}}{dt} \tag{2.4}$$

which, for the statically stable case $\rho^{(2)}/\rho^{(1)} < 1$, amounts to a balancing flow of smaller magnitude but in phase with the unsteady flow in the upper fluid. Actually, one would expect a modification of the flow in the lower fluid with a more complicated phase relationship due to viscous shearing action.

On the other hand, because we assumed that no free currents flow in any of the two fluids, Ampere's law requires the magnetic field **H** to be curl-free and thus represented as the gradient of the magnetic scalar potential $\psi(x, y, z; t)$ as

$$\mathbf{H}^{(j)} = H_0\mathbf{e}_x + H_*^{(j)}\mathbf{e}_z - \nabla\psi^{(j)}, \qquad j = 1, 2 \tag{2.5}$$

where $\mathbf{e}_x$ and $\mathbf{e}_z$ are unit vectors along the x and z directions, respectively.

Since each region has a uniform magnetic permeability, Gauss' law requires the magnetic scalar potential to obey Laplace's equation. Thus the basic equations governing the hydrodynamic and the magnetic field potentials are

$$\nabla^2\varphi^{(1)} = \nabla^2\psi^{(1)} = 0, \qquad -h_1 < z < \eta \tag{2.6}$$

$$\nabla^2\varphi^{(2)} = \nabla^2\psi^{(2)} = 0, \qquad \eta < z < h_2 \tag{2.7}$$

with

$$\left(\frac{\partial\varphi^{(1)}}{\partial z}\right)_{z=-h_1} = \left(\frac{\partial\psi^{(1)}}{\partial x}\right)_{z=-h_1} = 0 \tag{2.8}$$

$$\left(\frac{\partial\varphi^{(2)}}{\partial z}\right)_{z=h_2} = \left(\frac{\partial\psi^{(2)}}{\partial x}\right)_{z=h_2} = 0 \tag{2.9}$$

3. THE SOLUTIONS OF THE PERTURBATION EQUATIONS

We now assume the flow to be perturbed by a disturbance of sufficiently small magnitude, so that we may consider the linearized version of (2.1) for the disturbance pressure $\pi_1^{(j)}$ as

$$-\pi_1^{(j)} = \rho^{(j)}U^{(j)}(t)\frac{\partial \varphi_1^{(j)}}{\partial x} + \rho^{(j)}g\eta + \rho^{(j)}\frac{\partial \varphi_1^{(j)}}{\partial t} \tag{3.1}$$

As it is defined, η is the displacement of the interface from $z = 0$. We assume that the disturbance may be expressed in terms of its normal modes in the (x, y) directions and so express any function $\chi_1(x, y, z; t)$ as

$$\chi_1(x, y, z; t) = \chi(z, t)\exp[i(xk_x + yk_y)] \tag{3.2}$$

As we consider the fluid to be of finite depth, the appropriate solutions of the equations of motions are then (for $j = 1, 2$)

$$\varphi_1^{(j)} = A^{(j)}(t)\cosh k[z + (-1)^{j+1}h_j]\exp[i(xk_x + yk_y)] \tag{3.3}$$

$$\psi_1^{(j)} = B^{(j)}(t)\sinh k[z + (-1)^{j+1}h_j]\exp[i(xk_x + yk_y)] \tag{3.4}$$

where $k^2 = k_x^2 + k_y^2$, and $A^{(j)}(t)$ and $B^{(j)}(t)$ are time-dependent integration constants to be determined from the appropriate boundary conditions.

Since the transfer of mass across the interface represents a transformation of the fluid from one phase to another, there is invariably a latent heat associated with the phase change. Essentially, through this interfacial coupling between the mass transfer and the release of latent heat the motion of fluids is influenced by the thermal effects. Therefore, when there is significant mass transfer across the interface, the transfer of heat in the fluid has to be taken into consideration.

Based on a careful investigation of the results obtained by Hsieh (1972), it is reasonable to deduce that the amount of released latent heat depends mainly on the instantaneous position of the interface. More specifically, let us express the interface by

$$S(x, y, z; t) = z - \eta(x, y; t) \tag{3.5}$$

We propose that the interfacial condition for energy transfer can be expressed as

$$L\rho^{(j)}\left(\frac{\partial S}{\partial t} + \mathbf{V}^{(j)}\cdot\nabla S\right) = \theta(\eta) \tag{3.6}$$

where L is the latent heat released when the fluid is transformed from phase 1 to phase 2. The expression $\theta(\eta)$ essentially represents the net heat flux from the interface when such a phase transformation is taken place. In general (Hsieh, 1972), the heat flux has to be determined from the equations governing the heat transfer in the fluids. This is completely determined by the coupling between the dynamics and the thermal exchanges in the entire flow region. In this simplified version, the assumption is that θ is simply a function of

η, and moreover, θ is to be determined from the heat exchange relations in the equilibrium state.

The heat fluxes in the $+z$ direction in regions 1 and 2 are $k^{(1)}(T_1 - T_0)/h_1$ and $k^{(2)}(T_0 - T_2)/h_2$, respectively. As in Hsieh (1978), we denote

$$\theta(z) = \frac{k^{(2)}(T_0 - T_2)}{h_2 - z} - \frac{k^{(1)}(T_1 - T_2)}{h_1 + z} \tag{3.7}$$

It is clear that $\theta(z)$ represents the net heat flux from the interface into the fluid regions. Since it is an equilibrium state, we have

$$\theta(0) = 0 \tag{3.8}$$

Now, we can expand $\theta(\eta)$ about $\eta = 0$ by

$$\theta(\eta) = \theta'(0)\eta + \frac{1}{2}\theta''(0)\eta^2 + \cdots \tag{3.9}$$

From (3.7), we obtain

$$\theta'(0) = G\left(\frac{1}{h_1} + \frac{1}{h_2}\right) \tag{3.10}$$

where

$$G = \frac{k^{(2)}(T_0 - T_2)}{h_2} = \frac{k^{(1)}(T_1 - T_0)}{h_1}$$

which is the equilibrium heat flux from the plane $z = -h_1$ to $z = h_2$. Therefore, we may express $A^{(j)}(t)$ in terms of $\eta(t)$ as

$$A^{(j)}(t) = \frac{(-)^{j+1}}{k \sinh kh_j}\left(\frac{d}{dt} + ik_xU^{(j)}(t) + \frac{\alpha}{\rho^{(j)}}\right)\eta \tag{3.11}$$

where

$$\alpha = \frac{G}{L}\left(\frac{1}{h_1} + \frac{1}{h_2}\right)$$

The physical system used before is considered as a liquid–vapor system. Since the vapor phase is usually hotter than the liquid phase, α is always positive. If fluid 2 is a liquid and fluid 1 is a vapor, then L is positive and G is also positive, since $T_1 > T_0 > T_2$. If fluid 1 is a liquid and fluid 2 is a vapor, then L and G are both negative.

By the use of (3.3) and (3.11), we may now write from (3.1) the jump in perturbation pressure across the interface as

$$\frac{k(\pi_1^{(2)} - \pi_1^{(1)})}{\rho^{(1)} \coth kh_1 + \rho^{(2)} \coth kh_2}$$
$$= \frac{d^2\eta}{dt^2} + \left\{\alpha\left(\frac{\beta^{(1)}}{\rho^{(1)}} + \frac{\beta^{(2)}}{\rho^{(2)}}\right) + 2ik_x[\beta^{(1)}U^{(1)}(t) + \beta^{(2)}U^{(2)}(t)]\right\}\frac{d\eta}{dt}$$
$$+ \left\{\frac{kg(\rho^{(1)} - \rho^{(2)})}{\rho^{(1)} \coth kh_1 + \rho^{(2)} \coth kh_1} - k_x^2[\beta^{(1)}U^{(1)^2}(t) + \beta^{(2)}U^{(2)^2}(t)]\right.$$
$$+ ik_x\left[\beta^{(1)}\frac{d}{dt}U^{(1)}(t) + \beta^{(2)}\frac{d}{dt}U^{(2)}(t)\right]$$
$$\left. + i\alpha k_x\left(\frac{\beta^{(1)}U^{(1)}(t)}{\rho^{(1)}} + \frac{\beta^{(2)}U^{(2)}(t)}{\rho^{(2)}}\right)\right\}\eta \tag{3.12}$$

where

$$\beta^{(j)} = \frac{\rho^{(j)} \coth kh_j}{\rho^{(1)} \coth kh_1 + \rho^{(2)} \coth kh_2}$$

For the magnetic part, following the boundary conditions as given by Rosensweig (1985), one gets

$$B^{(j)}(t) = \frac{(\mu^{(j+1)} - \mu^{(j)})[ik_x H_0 \sinh kh_{j+1} + (-1)^{j+1}H_*^{(j)} \cosh kh_{j+1}]}{k(\mu^{(1)} \cosh kh_1 \sinh kh_2 + \mu^{(2)} \sinh kh_1 \cosh kh_2)}\eta \tag{3.13}$$

The jump in the perturbed surface pressure is allowed due to the effect of surface tension, so that

$$\pi_1^{(2)} - \frac{\mu^{(2)}}{2}(H_n^{(2)^2} - H_t^{(2)^2}) - \pi_1^{(1)} + \frac{\mu^{(1)}}{2}(H_n^{(1)^2} - H_t^{(1)^2}) = -k^2\sigma\eta \tag{3.14}$$

where σ is the surface tension coefficient,

$$H_n^2 = (\mathbf{n}\cdot\mathbf{H})^2 \qquad \text{and} \qquad H_t^2 = (\mathbf{n} \times \mathbf{H})^2$$

Here $\mathbf{n}$ is the unit vector normal to the interface. The final equation of $\eta(t)$ is then

$$\frac{d^2\eta}{dt^2} + \left\{\alpha\left(\frac{\beta^{(1)}}{\rho^{(1)}} + \frac{\beta^{(2)}}{\rho^{(2)}}\right) + 2ik_x[\beta^{(1)}U^{(1)}(t) + \beta^{(2)}U^{(2)}(t)]\right\}\frac{d\eta}{dt}$$
$$+ \left\{\frac{1}{\rho^{(1)} \coth kh_1 + \rho^{(2)} \coth kh_2}\left[k(\rho^{(1)} - \rho^{(2)})g + k^3\sigma\right.\right.$$

$$+ \frac{k_x^2(\mu^{(2)} - \mu^{(1)})^2 H_0^2}{\mu^{(1)} \coth kh_1 + \mu^{(2)} \coth kh_2}$$

$$- \left. \frac{k_x^2(\mu^{(2)} - \mu^{(1)})^2 H_{**}^2}{\mu^{(1)}\mu^{(2)}(\mu^{(1)} \tanh kh_1 + \mu^{(2)} \tanh kh_2)} \right]$$

$$+ ik_x\left[\beta^{(1)} \frac{d}{dt} U^{(1)}(t) + \beta^{(2)} \frac{d}{dt} U^{(2)}(t)\right]$$

$$\left. + i\alpha k_x\left(\frac{\beta^{(1)}U^{(1)}(t)}{\rho^{(1)}} + \frac{\beta^{(2)}U^{(2)}(t)}{\rho^{(2)}}\right)\right\}\eta = 0 \tag{3.15}$$

where $\mu^{(1)}H_*^{(1)} = \mu^{(2)}H_*^{(2)} = H_{**}$.

Equation (3.15) is a second-order ordinary differential equation with time-dependent coefficients. The nature of its solution will determine the stability or instability of the considered system. For thick layers ($h_{1,2} \to \infty$), an equation similar to (3.15) was first obtained, in the absence of mass and heat transfer as well the magnetic field, by Kelly (1965) and then by Roberts (1973) in the presence of a tangential magnetic field. Roberts' discussion of the effect of the unsteady Kelvin–Helmholtz flow is therefore applicable, with slight changes, to the considered system.

The analysis of equation (3.15) is classified into two categories as in the following two sections.

4. THE STABILITY OF STEADY KELVIN–HELMHOLTZ FLOW

When the velocities of the basic flow are constants, equation (3.15) becomes

$$\frac{d^2\eta}{dt^2} + \left[\alpha\left(\frac{\beta^{(1)}}{\rho^{(1)}} + \frac{\beta^{(2)}}{\rho^{(2)}}\right) + 2ik_x(\beta^{(1)}U^{(1)} + \beta^{(2)}U^{(2)})\right]\frac{d\eta}{dt}$$

$$+ \left\{\frac{1}{\rho^{(1)} \coth kh_1 + \rho^{(2)} \coth kh_2}\left[k(\rho^{(1)} - \rho^{(2)})g + k^3\sigma\right.\right.$$

$$+ \frac{k_x^2(\mu^{(2)} - \mu^{(1)})^2 H_0^2}{\mu^{(1)} \coth kh_1 + \mu^{(2)} \coth kh_2}$$

$$- \left. \frac{k_x^2(\mu^{(2)} - \mu^{(1)})^2 H_{**}^2}{\mu^{(1)}\mu^{(2)}(\mu^{(1)} \tanh kh_1 + \mu^{(2)} \tanh kh_2)} \right]$$

$$\left. + i\alpha k_x\left(\frac{\beta^{(1)}U^{(1)}}{\rho^{(1)}} + \frac{\beta^{(2)}U^{(2)}}{\rho^{(2)}}\right)\right\}\eta = 0 \tag{4.1}$$

Equation (4.1) was first obtained, in the absence of mass and heat transfer, by Rosensweig (1979). In the absence of the magnetic field, equation (4.1) reduces to that obtained by Hsieh (1978).

The stability criterion is based on the condition that $\eta(t)$ is a bounded function. Thus, the critical stability condition becomes

$$J \equiv \frac{1}{\rho^{(1)} \coth kh_1 + \rho^{(2)} \coth kh_2} \left[k(\rho^{(1)} - \rho^{(2)})g + k^3\sigma + \frac{k_x^2(\mu^{(2)} - \mu^{(1)})^2 H_0^2}{\mu^{(1)} \coth kh_1 + \mu^{(2)} \coth kh_2} - \frac{k_z^2(\mu^{(2)} - \mu^{(1)})^2 H_{**}^2}{\mu^{(1)}\mu^{(2)}(\mu^{(1)} \tanh kh_1 + \mu^{(2)} \tanh kh_2)} \right] - k_x^2(U^{(1)} - U^{(2)})^2\beta^{(1)}\beta^{(2)}\left[1 + \left(\frac{\rho^{(2)} - \rho^{(1)}}{\rho^{(1)}\beta^{(1)} + \rho^{(2)}\beta^{(2)}}\right)^2 \beta^{(1)}\beta^{(2)}\right] = 0 \tag{4.2}$$

The system is stable if $J \geq 0$ and unstable if $J < 0$. The expression of J differs from that of the classical Kelvin–Helmholtz problem by the additional last term.

From the above relation, we find that the critical condition of stability, in the presence of mass and heat transfer, is independent of the mass and heat transfer coefficient, but it differs from that in the absence of mass and heat transfer by the additional last term. On the other hand, when $U^{(1)} = U^{(2)}$ (Rayleigh–Taylor problem) or $\rho^{(1)} = \rho^{(2)}$, this term disappears in (4.2). This means that the mass and heat transfer has no effect on the problem. Therefore, the last term in (4.2) arises due to the effect of mass and heat transfer. Since this term is also positive, we conclude the destabilizing influence of the mass and heat transfer on the horizontal wave motion. The same role is also played by the transverse magnetic field. In contrast, the longitudinal field has a stabilizing effect and this can be used to suppress the destabilizing influence of the parameter α.

5. THE STABILITY ANALYSIS OF AN OSCILLATORY KELVIN–HELMHOLTZ FLOW

Let us now consider the time-dependent problem described by equation (3.15). From the preliminary remarks concerning parametric resonance, we suspect that a resonance might occur when the flow velocities vary periodically with time and then consequently the wave speed is being modified continuously.

We use the method of multiple scales (Nayfeh, 1985) in order to obtain an approximate solution and analyze the stability criteria for the problem. In accordance with this scheme, a small dimensionless parameter ϵ is needed. Introducing this small parameter as

$$U^{(j)}(t) = \epsilon U_0^{(j)} \text{ cost } \omega t \tag{5.1}$$

where $U_0^{(j)}$ is some constant having the dimension of velocity, we have that equation (3.15) then becomes

$$\frac{d^2\eta}{dt^2} + (a_1 + i\epsilon b_1 \cos \omega t)\frac{d\eta}{dt} + [a_2 + i\epsilon(b_2 \cos \omega t + b_3 \sin \omega t)]\eta = 0 \tag{5.2}$$

where

$$a_1 = \alpha\left(\frac{\beta^{(1)}}{\rho^{(1)}} + \frac{\beta^{(2)}}{\rho^{(2)}}\right)$$

$$b_1 = 2k_x(\beta^{(1)}U_0^{(1)} + \beta^{(2)}U_0^{(2)})$$

$$a_2 = \frac{1}{\rho^{(1)} \coth kh_1 + \rho^{(2)} \coth kh_2}\left[kg(\rho^{(1)} - \rho^{(2)}) + k^3\sigma + \frac{k_x^2(\mu^{(2)} - \mu^{(1)})^2 H_0^2}{\mu^{(1)} \coth kh_1 + \mu^{(2)} \coth kh_2} - \frac{k_x^2(\mu^{(2)} - \mu^{(1)})^2 H_{**}^2}{\mu^{(1)}\mu^{(2)}(\mu^{(1)} \tanh kh_1 + \mu^{(2)} \tanh kh_2)}\right]$$

$$b_2 = \alpha k_x\left(\frac{\beta^{(1)}U_0^{(1)}}{\rho^{(1)}} + \frac{\beta^{(2)}U_0^{(2)}}{\rho^{(2)}}\right)$$

$$b_3 = -\omega k_x(\beta^{(1)}U_0^{(1)} + \beta^{(2)}U_0^{(2)})$$

One assumes that the solution of equation (5.2) can be represented by

$$\eta(t, \epsilon) = \eta_0(T_0, T_1, T_2) + \epsilon\eta_1(T_0, T_1, T_2) + \epsilon^2\eta_2(T_0, T_1, T_2) + \cdots \tag{5.3}$$

Then we insert the above perturbation solution (5.3) into Mathieu's equation (5.2), transform the time derivatives, and collect coefficients of each power of ϵ. These equations must hold independently because sequences of ϵ are linearly independent. The resulting equations can be solved successively. Thus we have

$$\epsilon^0: \quad (D_0^2 + a_1 D_0 + a_2)\eta_0 = 0 \tag{5.4}$$

$$\begin{aligned}\epsilon^1: \quad &(D_0^2 + a_1 D_0 + a_2)\eta_1 \\ &= -2D_0 D_1 \eta_0 - a_1 D_1 \eta_0 - i b_1 \cos \omega T_0 \, D_0 \eta_0 - i(b_2 \cos \omega T_0 \\ &\quad + b_3 \sin \omega T_0)\eta_0 \end{aligned} \tag{5.5}$$

$$\begin{aligned}\epsilon^2: \quad &(D_0^2 + a_1 D_0 + a_2)\eta_2 \\ &= -2D_0 D_1 \eta_1 - (D_1^2 + 2D_0 D_2)\eta_0 - a_1(D_1\eta_1 + D_2\eta_0) \\ &\quad - i b_1 \cos \omega T_0 \,(D_0\eta_1 + D_1\eta_0) - i(b_2 \cos \omega T_0 + b_3 \sin \omega T_0)\eta_1 \end{aligned} \tag{5.6}$$

where $D_n \equiv \partial/\partial T_n$.

With this approach it is convenient to write the solution of equation (5.4) in the form

$$\eta_0(T_0, T_1, T_2) = \gamma(T_1, T_2) \exp \Omega T_0 + \text{c.c.} \tag{5.7}$$

where γ is an unknown complex function of T_1 and T_2, c.c. represents the complex conjugate of the preceding terms, and Ω is a complex frequency given by

$$\Omega^2 + a_1\Omega + a_2 = 0 \tag{5.8}$$

If $\Omega = \Omega_r + i\Omega_i$ with real Ω_r and Ω_i, then we get $\Omega_r = -\frac{1}{2}a_1$ and $\Omega_i^2 = a_2 - \frac{1}{4}a_1^2$. Thus the stability conditions in the zeroth order in ϵ are

$$a_1 > 0 \qquad \text{and} \qquad a_2 - \frac{1}{4}a_1^2 \geq 0 \tag{5.9}$$

The first condition is automatically satisfied, while the second one represents the stability criterion in the case of Rayleigh–Taylor instability.

The solution of equation (5.5) can be obtained from knowledge of the zeroth order in ϵ. Therefore, substitution from (5.7) into (5.5) yields

$$\begin{aligned}&(D_0^2 + a_1 D_0 + a_2)\eta_1 \\ &\quad = -2i\Omega_i D_1 \gamma \exp \Omega T_0 \\ &\qquad - \frac{1}{2}\left[(b_3 - b_1\Omega_i) + i\left(b_2 - \frac{1}{2}a_1 b_1\right)\right]\gamma \exp(\Omega + i\omega)T_0 \\ &\qquad + \frac{1}{2}\left[(b_3 + b_1\Omega_i) - i\left(b_2 - \frac{1}{2}a_1 b_1\right)\right]\gamma \exp(\Omega - i\omega)T_0 + \text{c.c.}\end{aligned} \tag{5.10}$$

Equation (5.10) contains nonhomogeneous terms. The uniform solution is required to eliminate the secular terms. This elimination introduces the solvability conditions which correspond to the terms containing the factor exp ΩT_0. Thus in order to analyze the solution of equation (5.10), we need to distinguish between two cases: the nonresonance case, when the frequency ω is not near Ω_i; and the resonance case, which arises when the frequency ω is approaching the frequency Ω_i or $2\Omega_i$.

5.1. The Case of ω Not Near $2\Omega_i$

In order to obtain a uniformly valid expansion, the coefficient of the factor $\exp(\Omega T_0)$ in equation (5.10) must vanish. Thus, we get

$$D_1\gamma(T_1, T_2) = 0 \tag{5.11}$$

It follows that γ must be independent of T_1, which gives $\gamma = \gamma(T_2)$. Therefore, the first-order uniformly valid expansion is given by

$$\eta_1 = \frac{1}{2\omega}\left(\frac{(b_3 - b_1\Omega_i) + i(b_2 - \frac{1}{2}a_1b_1)}{2\Omega_i + \omega}\gamma\exp(\Omega + i\omega)T_0\right.$$
$$\left. + \frac{(b_3 + b_1\Omega_i) - i(b_2 - \frac{1}{2}a_1b_1)}{2\Omega_i + \omega}\gamma\exp(\Omega - i\omega)T_0\right) + \text{c.c.} \tag{5.12}$$

Substituting the solution of η_0 and η_1 into equation (5.6) of the second order in ϵ, we obtain

$$(D_0^2 + a_1D_0 + a_2)\eta_2$$
$$= \left(-(D_1^2 + 2i\Omega_iD_2) - \frac{i}{2}\left\{\left(b_2 - \frac{1}{2}a_1b_1\right) + i[b_1(\Omega_i + \omega) + b_3]\right\}A\right.$$
$$\left. - \frac{i}{2}\left\{\left(b_2 - \frac{1}{2}a_1b_1\right) + i[b_1(\Omega_i - \omega) - b_3]\right\}B\right)\gamma\exp\Omega T_0$$
$$- \frac{i}{2}[b_1 + 4(\Omega_i + \omega)A]D_1\gamma\exp(\Omega + i\omega)T_0$$
$$- \frac{i}{2}[b_1 + 4(\Omega_i - \omega)B]D_1\gamma\exp(\Omega - i\omega)T_0$$
$$- \frac{i}{2}\left\{\left(b_2 - \frac{1}{2}a_1b_1\right) + i[b_1(\Omega_i + \omega) - b_3]\right\}A\gamma\exp(\Omega + 2i\omega)T_0$$
$$- \frac{i}{2}\left\{\left(b_2 - \frac{1}{2}a_1b_1\right) + i[b_1(\Omega_i - \omega) + b_3]\right\}B\gamma\exp(\Omega - 2i\omega)T_0 + \text{c.c.} \tag{5.13}$$

where

$$A = C_1 + iE_1, \qquad B = C_2 + iE_2$$

$$C_j = \frac{1}{2\omega[2\Omega_i + (-)^{j-1}\omega]}(b_3 + (-)^j\Omega_i)$$

$$E_j = \frac{(-)^{j-1}}{2\omega[2\Omega_i + (-)^{j-1}\omega]}\left(b_2 - \frac{1}{2}a_1b_1\right), \qquad j = 1, 2$$

In order to determine a particular solution of the above equation, we need to distinguish between the case of ω being away from Ω_i and the case of ω approaching Ω_i.

(I) *The case of* ω *not near* Ω_i. The vanishing of the terms that produce secular terms in equation (5.13) yields

$$D_2\gamma + (P_1 + iP_2)\gamma = 0 \tag{5.14}$$

where (5.11) has been used. The real coefficients P_1 and P_2 are given by

$$P_1 = \frac{1}{4\Omega_i}\left\{\left(b_2 - \frac{1}{2}a_1b_1\right)(C_1 + C_2) - [b_1(\Omega_i + \omega) + b_3]E_1 - [b_1(\Omega_i - \omega) - b_3]E_2\right\} \tag{5.15}$$

$$P_2 = \frac{1}{4\Omega_i}\left\{\left(b_2 - \frac{1}{2}a_1b_1\right)(E_1 + E_2) + [b_1(\Omega_i + \omega) + b_3]C_1 + [b_1(\Omega_i - \omega) - b_3]C_2\right\} \tag{5.16}$$

Equation (5.14) has the following solution:

$$\gamma(T_2) = \hat{\gamma}e^{-(P_1+iP_2)T_2}$$

with a constant $\hat{\gamma}$.

Therefore, stability is present in the nonresonance case (where ω is away from $2\Omega_i$ and ω away from Ω_i) when

$$P_1 > 0 \tag{5.17}$$

(II) *The case of* ω *near* Ω_i. We express the nearness of ω to Ω_i by introducing the detuning parameter δ defined as

$$\omega = \Omega_i + \epsilon^2\delta \tag{5.18}$$

and hence

$$-i(\Omega_i - 2\omega)T_0 = i\Omega_i T_0 + 2i\delta T_2 \tag{5.19}$$

The elimination of the terms that produce secular terms in equation (5.13) implies

$$D_2\gamma + (P_1 + iP_2)\gamma + (Q_1 + iQ_2)\bar{\gamma} \exp 2i\delta T_0 = 0 \tag{5.20}$$

where the solvability condition (5.11) is used. The coefficients P_1 and P_2 are defined by (5.15) and (5.16), while the coefficients Q_1 and Q_2 are given by

$$Q_1 = -\frac{1}{4\Omega_i}\left\{\left(b_2 - \frac{1}{2}a_1 b_1\right)C_2 - [b_1(\Omega_i - \omega) + b_3]E_2\right\} \tag{5.21}$$

$$Q_2 = \frac{1}{4\Omega_i}\left\{\left(b_2 - \frac{1}{2}a_1 b_1\right)E_2 + [b_1(\Omega_i - \omega) + b_3]C_2\right\} \tag{5.22}$$

To solve equation (5.20), we let

$$\gamma(T_2) = [f_1(T_2) + if_2(T_2)]\exp i\delta T_0 \tag{5.23}$$

where $f_1(T_0)$ and $f_2(T_0)$ are real functions of T_2. Inserting the solution (5.23) into equation (5.20), separating the real and imaginary parts, we obtain the following equations that govern $f_1(T_0)$ and $f_2(T_0)$:

$$[D_2 + (P_1 + Q_1)]f_1 + (Q_2 - P_2 - \delta)f_2 = 0 \tag{5.24}$$

$$[D_2 + (P_1 - Q_1)]f_1 + (P_2 + Q_2 + \delta)f_2 = 0 \tag{5.25}$$

If the pair of equations (5.24) and (5.25) have the solutions

$$f_1(T_2) = \hat{f}_1 \exp \nu T_0 + \text{c.c.} \tag{5.26}$$

$$f_2(T_2) = \hat{f}_2 \exp \nu T_0 + \text{c.c.} \tag{5.27}$$

with real constants $\hat{f}_1$ and $\hat{f}_2$, then the characteristic frequency ν is given by

$$\nu^2 + 2P_1\nu + [\delta^2 + 2P_2\delta + P_1^2 + P_2^2 - (Q_1^2 + Q_2^2)] = 0 \tag{5.28}$$

Therefore, and in view of the Hurwitz criteria, the stability requires the conditions

$$P_1 > 0 \tag{5.29}$$

$$\delta^2 + 2P_2\delta + (P_1^2 + P_2^2) - (Q_1^2 + Q_2^2) > 0 \tag{5.30}$$

The first condition is satisfied in the nonresonance case, while the second one can be written in the form

$$(\delta - \delta_1^*)(\delta - \delta_2^*) > 0 \tag{5.31}$$

which can be satisfied when

$$\delta > \delta_1^* \quad \text{and} \quad \delta < \delta_2^* \quad (\delta_1^* > \delta_2^*)$$

where

$$\delta_{1,2}^* = -p_2 \pm (Q_1^2 + Q_2^2 - P_1^2)^{1/2} \tag{5.32}$$

In view of (5.18), we can write the transition curves separating the stable region from the unstable region

$$\omega = \Omega_i + \epsilon^2[-p_2 + (Q_1^2 + Q_2^2 - P_1^2)^{1/2}] \tag{5.33}$$

and

$$\omega = \Omega_i + \epsilon^2[-P_2 - (Q_1^2 + Q_2^2 - P_1^2)^{1/2}] \tag{5.34}$$

or

$$\epsilon_1^2 = \frac{\omega - \Omega_i}{-p_2 + (Q_1^2 + Q_2^2 - P_1^2)^{1/2}} \tag{5.35}$$

and

$$\epsilon_2^2 = \frac{\omega - \Omega_i}{-p_2 - (Q_1^2 + Q_2^2 - P_1^2)^{1/2}} \tag{5.36}$$

5.2. The Case of ω Near $2\Omega_i$

Let us return to equation (5.10) and analyze it in the case of the frequency ω approaching twice the frequency Ω_i. In order to obtain a uniformly valid expansion in this version, we express the nearness of ω to $2\Omega_i$ by introducing the detuning parameter μ according to

$$\omega = 2\Omega_i + 2\epsilon\mu \tag{5.37}$$

and hence

$$-i(\Omega_i - \omega)T_0 = i\Omega_i T_0 + 2i\mu T_1 \tag{5.38}$$

Thus, the elimination of the secular terms from equation (5.10) gives the following solvability condition:

$$D_1\gamma - \frac{i}{4\Omega_i}\left[(b_1\Omega_i + b_3) - i\left(b_2 - \frac{1}{2}a_2b_1\right)\right]\bar{\gamma}\exp 2i\mu T_1 = 0 \tag{5.39}$$

The uniformly valid expansion for $\eta_1(T_0, T_1, T_2)$ in this case is

$$\eta_1 = \frac{(b_3 - b_1\Omega_i) + i\left(b_2 - \frac{1}{2}a_1b_1\right)}{2\omega(2\Omega_i + \omega)}\gamma(T_1, T_2)\exp(\Omega + i\omega)T_0 + \text{c.c.} \tag{5.40}$$

Substituting (5.7) and (5.40) into (5.6), we get

$$\begin{aligned}(D_0^2 &+ a_1D_0 + a_2)\eta_2 \\ &= \left(-(D_1^2 + 2i\Omega_iD_2) - \frac{i}{2}\left\{\left(b_2 - \frac{1}{2}a_1b_1\right) + i[b_1(\Omega_i + \omega) + b_3]\right\}A\right. \\ &\quad \left. - \frac{i}{2}\left\{\left(b_2 - \frac{1}{2}a_1b_1\right) + i[b_1(\Omega_i - \omega) - b_3]\right\}B\right)\gamma\exp\Omega T_0 \\ &\quad + \frac{i}{2}[b_1 + 4(\Omega_i - \omega)\bar{B}]D_1\bar{\gamma}\exp\Omega T_0\exp 2i\mu T_1 + \text{NST} + \text{c.c.}\end{aligned} \tag{5.41}$$

where NST stands for terms that do not produce secular terms. The elimination of the secular terms appearing in equation (5.41) gives

$$\begin{aligned}&\left(-(D_1^2 + 2i\Omega_iD_2) - \frac{i}{2}\left\{\left(b_2 - \frac{1}{2}a_1b_1\right) + i[b_1(\Omega_i + \omega) + b_3]\right\}A\right. \\ &\quad \left. - \frac{i}{2}\left\{\left(b_2 - \frac{1}{2}a_1b_1\right) + i[b_1(\Omega_i - \omega) - b_3]\right\}B\right)\gamma \\ &\quad + \frac{i}{2}[b_1 + 4(\Omega_i - \omega)\bar{B}]D_1\bar{\gamma}\exp 2i\mu T_1 = 0\end{aligned} \tag{5.42}$$

Combining the solvability condition (5.39) with the above one gives

$$D_2\gamma + (R_1 + iR_2)\gamma + \mu(S_1 + iS_2)\bar{\gamma}\exp 2i\mu T_1 = 0 \tag{5.43}$$

where

$$\begin{aligned}R_1 = -\frac{1}{8\Omega_i}&\left\{\left(b_2 - \frac{1}{2}a_1b_1\right)[b_1 + 4(\Omega_i - \omega)]C_2\right. \\ &\left. - 4(\Omega_i - \omega)(b_1\Omega_i + b_3)E_2\right\} + P_1\end{aligned} \tag{5.44}$$

$$R_2 = -\frac{1}{8\Omega_i}\left\{4\left(b_2 - \frac{1}{2}a_1b_1\right)(\Omega_i - \omega)E_2 + [b_1 + 4(\Omega_i - \omega)](b_1\Omega_i + b_3)C_2\right\} + P_2 \tag{5.45}$$

$$S_1 = \frac{1}{4\Omega_i^2}\left(b_2 - \frac{1}{2}a_1b_1\right) \tag{5.46}$$

$$S_2 = \frac{1}{4\Omega_i^2}(b_1\Omega_i + b_3) \tag{5.47}$$

Assume that equation (5.43) admits a nontrivial solution in the form

$$\gamma(T_1, T_2) = [u(T_2) + iv(T_2)] \exp i\mu T_1 + \text{c.c.} \tag{5.48}$$

with real functions u and v.

Substituting (5.48) into (5.43) and separating the real and imaginary parts, we get

$$[D_2 + (R_1 + \mu S_1)]u + (\mu S_2 - R_2)v = 0 \tag{5.49}$$

$$[D_2 + (R_1 - \mu S_1)]v + (R_2 + \mu S_2)u = 0 \tag{5.50}$$

Since the above equations are coupled linear differential equations of the first order, their solutions can be sought in the form

$$u(T_2) = \hat{u} \exp \lambda T_2 + \text{c.c.} \tag{5.51}$$

$$v(T_2) = \hat{v} \exp \lambda T_2 + \text{c.c.} \tag{5.52}$$

with real constants $\hat{u}$ and $\hat{v}$.

Inserting (5.51) and (5.52) into equations (5.49) and (5.50), for the nontrivial solutions of $\hat{u}$ and $\hat{v}$, we get the following dispersion relation:

$$\lambda^2 + 2R_1\lambda + [R_1^2 + R_2^2 - \mu^2(S_1^2 + S_2^2)] = 0 \tag{5.53}$$

Since the necessary and sufficient condition for stability is present when all roots of the above equation have negative real parts, stability occurs when

$$R_1 > 0 \tag{5.54}$$

$$R_1^2 + R_2^2 - \mu^2(S_1^2 + S_2^2) > 0 \tag{5.55}$$

Thus the transition curves separating the stable region from the unstable one correspond to

$$\mu^2 = \frac{R_2^2}{S_1^2 + S_2^2} \tag{5.56}$$

Hence, in view of (5.37) we obtain the transition curves in the form

$$\omega = 2\Omega_i + \frac{2R_2\epsilon}{(S_1^2 + S_2)^{1/2}} + O(\epsilon^2) + \cdots \tag{5.57}$$

and

$$\omega = 2\Omega_i - \frac{2R_2\epsilon}{(S_1^2 + S_2^2)^{1/2}} + O(\epsilon^2) + \cdots \tag{5.58}$$

In terms of the amplitude disturbance we get

$$\epsilon_1 = \frac{(\omega - 2\Omega_i)(S_1^2 + S_2^2)^{1/2}}{2R_2} \tag{5.59}$$

and

$$\epsilon_2 = \frac{(2\Omega_i - \omega)(S_1^2 + S_2^2)^{1/2}}{2R_2} \tag{5.60}$$

The values of δ and μ as described by equations (5.32) and (5.56) are the critical values of the disturbances. These critical values, which are known as the transition curves, separate the stable from the unstable region. According to Floquet's theory (Nayfeh, 1985), the region bounded by the two branches of the transition curves is unstable, while the area outside them is stable.

6. CONCLUSIONS

The linear ferrohydrodynamic Kelvin–Helmholtz flow on a horizontal interface between two inviscid fluids has been studied. The two fluids are enclosed between two horizontal rigid plates in parallel with the interface. The interface admits the presence of mass and heat transfer. The system is stressed by a uniform oblique magnetic field. In the stationary state, the fluids are streaming parallel to each other in an oscillatory manner. The analysis was undertaken principally by clarifying the coupling between the effect of mass and heat transfer and the periodic Kelvin–Helmholtz flow on the stability of the horizontal interface between two ferrodynamic inviscid fluids. All modes of disturbances are considered. The stability of the system is analytically discussed. We draw the following conclusions:

1. In the presentation of the problem, the effects of mass and heat transfer are revealed through a single parameter α. Thus, the correlation of the experimental data would be very greatly facilitated by this simplification.

2. The stability criterion in the absence of the periodicity of the streaming velocities is independent of the mass and heat transfer coefficient α, but differs

from that of the same problem without mass and heat transfer. Generally, it is found that the mass and heat transfer coefficient tends to destabilize the system. The same role is played by the transverse magnetic field, while the longitudinal one has a stabilizing influence on the interface. The latter can be used to suppress the influence of the parameter α.

3. The parametric excitation of the oscillatory streaming results in a second-order ordinary linear differential equation with periodic coefficients. It has been solved by means of the multiple-time-scales scheme. The analysis reveals the existence of both resonant and nonresonant cases. The resonance modes appear due to oscillatory Kelvin–Helmholtz streaming. The transition curves are obtained. According to the Floquet theory, the region bounded by the two branches of the transition curves is unstable, while the area outside them is stable.

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